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A non-confocal generator of involutive systems and Levy hierarchy

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Abstract. In this paper, a new non-confocal generator of involutive systems is obtained, and the relation between it and Levy equation hierarchy is discussed. Furthermore, the representation of the solution for the Levy hierarchy is given.

1. Introduction

To find a new integrable system is an important subject [1, 2] in soliton theory and integrability theory. However, whether a Hamiltonian system is completely integrable depends on whether the N-involutive system exists or not; almost all involutive systems already acquired by nonlinearized eigenvalue problems [3] are reduced to the so-called confocal involutive systems [4–14], whose generators are

$$T_k = \sum_{\substack{j=1\\ j\neq k}}^{N} \frac{(p_k q_j - p_j q_k)^2}{\alpha_k - \alpha_j} \qquad k = 1, 2, \ldots, N.$$

A natural problem is to find non-confocal generators, so that essential new integrable system can be obtained.

In this paper, a new generator of the non-confocal involutive system $\tilde{G}_k$ is found:

$$\tilde{G}_k = \sum_{\substack{j=1\\ j\neq k}}^{N} \frac{\lambda_k^2 p_k^2 q_j^2 + \lambda_j^2 p_j^2 q_k^2 - (\lambda_k^2 + \lambda_j^2) p_k p_j q_k q_j}{\lambda_k^2 - \lambda_j^2} + \frac{\lambda_j p_j^2 q_k^2 - \lambda_k p_k^2 q_j^2}{\lambda_k + \lambda_j}. \tag{1.1}$$

We prove that the Hamiltonian system $(R^{2N}, \mathrm{d}P \wedge \mathrm{d}Q = \Sigma_{j=1}^{N} \mathrm{d}p_j \wedge \mathrm{d}q_j, H)$ is completely integrable in the Liouville sense, where

$$H = \tfrac{1}{2}(\langle A^2 P, Q\rangle + \langle P, Q\rangle^2 - \langle AQ, Q\rangle\langle P, Q\rangle - \langle AP, P\rangle). \tag{1.2}$$

$A = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N)$, $\langle\cdot,\cdot\rangle$ are a standard inner product in R^N, $Q = (q_1, q_2, \ldots, q_N)^{\mathrm{T}}$, $P = (p_1, p_2, \ldots, p_N)^{\mathrm{T}}$. Furthermore, the relation between the completely integrable Hamiltonian system $(R^{2N}, \mathrm{d}P \wedge \mathrm{d}Q, H)$ and Levy equation hierarchy [15, 16] is discussed; the solutions of the Levy hierarchy are generated by the solutions of the completely integrable systems.

2. The non-confocal generator and the finite-dimensional complete integrable Hamiltonian systems [17]

Let $P = (p_1, p_2, \ldots, p_N)^{\mathrm{T}}$ and $Q = (q_1, q_2, \ldots, q_N)^{\mathrm{T}}$ be the basic coordinate functions in R^{2N}. The Poisson bracket of two smooth functions F_1 and F_2 in the symplectic space $(R^{2N}, \mathrm{d}P \wedge \mathrm{d}Q = \Sigma_{j=1}^{N} \mathrm{d}p_j \wedge \mathrm{d}q_j)$ is defined as [17]

$$(F_1, F_2) = \sum_{j=1}^{N} \frac{\partial F_1}{\partial q_j}\frac{\partial F_2}{\partial p_j} - \frac{\partial F_1}{\partial p_j}\frac{\partial F_2}{\partial q_j} \tag{2.1}$$

which satisfies the Leibnitz rule:

$$(F_1F_2, F_3) = F_1(F_2, F_3) + F_2(F_1, F_3)$$

where F_3 is the smooth function in the symplectic space $(R^{2N}, \mathrm{d}P \wedge \mathrm{d}Q)$. F_1 and F_2 is called an involution if $(F_1, F_2) = 0$.

The Hamiltonian canonical equation of the smooth function F in the symplectic space $(R^{2N}, \mathrm{d}P \wedge \mathrm{d}Q)$ is defined as [17]

$$\left.\begin{aligned} p_{jt} &= (p_j, F) = -\partial F/\partial q_j \\ q_{jt} &= (q_j, F) = \partial F/\partial p_j \end{aligned}\right\} \qquad j = 1, 2, \ldots, N. \tag{2.2}$$

Theorem 2.1. $(\tilde{G}_n, \tilde{G}_k) = 0, \quad n, k = 1, 2, \ldots, N.$

Proof. Evidently $(\tilde{G}_k, \tilde{G}_k) = 0$, let

$$n \neq k \qquad A_{kj} = \lambda_j p_j^2 q_k^2 - \lambda_k p_k^2 q_j^2$$

$$B_{kj} = \lambda_k^2 p_k^2 q_j^2 + \lambda_j^2 q_k^2 p_j^2 - (\lambda_k^2 + \lambda_j^2) p_k q_j q_k p_j$$

$$\sum_n{}' = \sum_{\substack{i=1\\ i\neq n}}^{N} \qquad \sum_k{}' = \sum_{\substack{j=1\\ j\neq k}}^{N} \qquad \sum_j{}'' = \sum_{\substack{j=1\\ j\neq k,n}}^{N} \qquad \sum_i = \sum_{\substack{i=1\\ i\neq n,k}}^{N}$$

then

$$(B_{kn}, B_{nk}) = 0 \qquad (A_{kn}, A_{nk}) = 0$$

and

$$\begin{aligned}(\tilde{G}_n, \tilde{G}_k) &= \left(\sum_n{}' \left(\frac{B_{ni}}{\lambda_n^2 - \lambda_i^2} + \frac{A_{ni}}{\lambda_n + \lambda_i}\right), \sum_k{}' \left(\frac{B_{kj}}{\lambda_k^2 - \lambda_j^2} + \frac{A_{kj}}{\lambda_k + \lambda_j}\right)\right) \\ &= \left(\sum_n{}' \frac{B_{ni}}{\lambda_n^2 - \lambda_i^2}, \sum_k{}' \frac{B_{kj}}{\lambda_k^2 - \lambda_j^2}\right) + \left(\sum_n{}' \frac{B_{ni}}{\lambda_n^2 - \lambda_i^2} - \sum_k{}' \frac{A_{kj}}{\lambda_k + \lambda_j}\right) \\ &\quad + \left(\sum_n{}' \frac{A_{ni}}{\lambda_n + \lambda_i}, \sum_k{}' \frac{A_{kj}}{\lambda_k + \lambda_j}\right) + \left(\sum_n{}' \frac{A_{ni}}{\lambda_n + \lambda_i}, \sum_k{}' \frac{B_{kj}}{\lambda_k^2 - \lambda_j^2}\right).\end{aligned}$$

Through direct calculation, we have

$$\begin{aligned}\left(\sum_n{}' \frac{B_{ni}}{\lambda_n^2 - \lambda_i^2}, \sum_k{}' \frac{B_{kj}}{\lambda_k^2 - \lambda_j^2}\right) &= \left(\sum_i{}'' \frac{B_{ni}}{\lambda_n^2 - \lambda_i^2}, \sum_j{}'' \frac{B_{kj}}{\lambda_k^2 - \lambda_j^2}\right) \\ &\quad + \left(\frac{B_{nk}}{\lambda_n^2 - \lambda_k^2}, \sum_j{}'' \frac{B_{kj}}{\lambda_k^2 - \lambda_j^2}\right) + \left(\sum_i{}'' \frac{B_{ni}}{\lambda_n^2 - \lambda_i^2}, \frac{B_{kn}}{\lambda_k^2 - \lambda_n^2}\right) = 0.\end{aligned}$$

Similarly

$$\left(\sum_n{}' \frac{B_{ni}}{\lambda_n^2 - \lambda_i^2}, \sum_k{}' \frac{A_{kj}}{\lambda_k + \lambda_j}\right) + \left(\sum_n{}' \frac{A_{ni}}{\lambda_n + \lambda_i}, \sum_k{}' \frac{A_{kj}}{\lambda_k + \lambda_j}\right) + \left(\sum_n{}' \frac{A_{ni}}{\lambda_n + \lambda_i}, \sum_k{}' \frac{B_{kj}}{\lambda_k^2 - \lambda_j^2}\right) = 0.$$

Thus $(\tilde{G}_n, \tilde{G}_k) = 0, \quad n, k = 1, 2, \ldots, N.$

Theorem 2.2. Set

$$Q_z(\xi, \eta) = \langle (zI - A^2)^{-1}\xi, \eta \rangle = \sum_{k=1}^{N} \frac{\xi_k \eta_k}{z - \lambda_k^2}$$

where I denotes the unit matrix, then the generalizing function of $\tilde{G}_k$ is

$$\sum_{k=1}^{N} \frac{\tilde{G}_k}{z - \lambda_k^2} = \begin{vmatrix} Q_z(AP, P) & Q_z(P, Q) \\ Q_z(A^2P, Q) & Q_z(AQ, Q) \end{vmatrix}$$

Proof. Since

$$Q_z(\xi, \eta) = \sum_{k=1}^{N} \frac{\xi_k \eta_k}{z - \lambda_k^2}$$

and

$$\frac{1}{(z-\lambda_k^2)(z-\lambda_j^2)} = \frac{1}{(z-\lambda_k^2)(\lambda_k^2-\lambda_j^2)} + \frac{1}{(z-\lambda_j^2)(\lambda_j^2-\lambda_k^2)}$$

the required result is obtained through direct calculation.

Theorem 2.3. Let

$$E_k = \lambda_k^2 p_k q_k - \lambda_k p_k^2 + \langle P, Q\rangle p_k q_k - \langle P, Q\rangle \lambda_k q_k^2 - \tilde{G}_k$$

then $\{E_k, k = 1, 2, \ldots, N\}$ is an N-involutive system.

Proof. Evidently $(E_n, E_n) = 0$, let $n \neq k$, then through direct calculation, we have

$$(-\lambda_n p_n^2, \langle P, Q\rangle p_k q_k) + (\langle P, Q\rangle p_n q_n, -\lambda_k p_k^2) + (-\lambda_n p_n^2, -\tilde{G}_k) + (-\tilde{G}_n, -\lambda_k p_k^2) = 0$$

$$(-\lambda_n p_n^2, -\langle P, Q\rangle \lambda_k q_k^2) + (-\langle P, Q\rangle \lambda_n q_n^2, -\lambda_k p_k^2) + (\lambda_n^2 p_n q_n, -\tilde{G}_k) + (-\tilde{G}_n, \lambda_k^2 p_k q_k) = 0$$

$$(\langle P, Q\rangle p_n q_n, -\tilde{G}_k) + (-\tilde{G}_n, \langle P, Q\rangle p_k q_k) = 0$$

$$\begin{aligned}(\langle P, Q\rangle p_n q_n, -\lambda_k \langle P, Q\rangle q_k^2) &+ (-\langle P, Q\rangle \lambda_n q_n^2, \langle P, Q\rangle p_k q_k) \\ &+ (-\lambda_n \langle P, Q\rangle q_n^2, -\tilde{G}_k) + (-\tilde{G}_n, -\lambda_k \langle P, Q\rangle q_k^2) = 0\end{aligned}$$

$$(\langle P, Q\rangle, p_k q_k) = 0 \qquad (p_n q_n, p_k q_k) = 0$$

$$(p_n^2, p_k q_k) = 0 \qquad (q_n^2, p_k q_k) = 0.$$

From theorem 2.1, $(\tilde{G}_n, \tilde{G}_k) = 0$, thus $(E_n, E_k) = 0, \quad n, k = 1, 2, \ldots, N.$

Theorem 2.4. The Hamiltonian system (2.3) by definition (1.2) is finite-dimensional and completely integrable in the Liouville sense.

$$\left.\begin{aligned} p_{jx} &= -\partial H / \partial q_j \\ q_{jx} &= \partial H / \partial p_j \end{aligned}\right\} \qquad j = 1, 2, \ldots, N. \tag{2.3}$$

Proof. From theorem 2.3, $(E_n, E_k)=0$, $n, k=1, 2, \ldots, N$, and through direct calculation, we have

$$(H, E_k)=0 \qquad k=1,2,\ldots,N.$$

The required result is obtained.

From theorem 2.2, let $|z|>\max\{|\lambda_1^2|, |\lambda_2^2|, \ldots, |\lambda_N^2|\}$, then

$$(z-\lambda_k^2)^{-1}=\sum_{m=0}^{\infty} z^{-(m+1)}\lambda_k^{2m}$$

so that

$$Q_z(\xi, \eta)=\sum_{m=0}^{\infty} z^{-(m+1)}\langle A^{2m}\xi, \eta\rangle$$

$$\sum_{k=1}^{N} \frac{\tilde{G}_k}{z-\lambda_k^2}=\sum_{m=0}^{\infty}\sum_{i+j=m-1} z^{-(m+1)}\begin{vmatrix}\langle A^{2i+1}P, P\rangle & \langle A^{2i}P, Q\rangle \\ \langle A^{2j+2}P, Q\rangle & \langle A^{2j+1}Q, Q\rangle\end{vmatrix}$$

thus

$$\sum_{k=1}^{N} \tilde{G}_k\lambda_k^{2m}=\sum_{j=1}^{m}\begin{vmatrix}\langle A^{2j-1}P, P\rangle & \langle A^{2j}P, Q\rangle \\ \langle A^{2m-2j}P, Q\rangle & \langle A^{2m-2j+1}Q, Q\rangle\end{vmatrix}.$$

Furthermore

$$\sum_{k=1}^{N} \tfrac{1}{2}E_k\lambda_k^{2m}=H_m \tag{2.4}$$

where

$$H_m=\tfrac{1}{2}\langle A^{2m+2}P, Q\rangle-\tfrac{1}{2}\langle A^{2m+1}P, P\rangle+\tfrac{1}{2}\langle P, Q\rangle\langle A^{2m}P, Q\rangle-\tfrac{1}{2}\langle P, Q\rangle\langle A^{2m+1}Q, Q\rangle$$
$$+\tfrac{1}{2}\sum_{j=1}^{m}\begin{vmatrix}\langle A^{2j}P, Q\rangle & \langle A^{2j-1}Q, Q\rangle \\ \langle A^{2m-2j+1}P, P\rangle & \langle A^{2m-2j}P, Q\rangle\end{vmatrix}. \tag{2.5}$$

By theorem 2.3, we have

$$(H_m, H_n)=\left(\sum_{k=1}^{N}\tfrac{1}{2}E_k\lambda_k^{2m}, \sum_{j=1}^{N}\tfrac{1}{2}E_j\lambda_j^{2n}\right)$$
$$=\sum_{k=1}^{N}\sum_{j=1}^{N}\tfrac{1}{4}\lambda_k^{2m}\lambda_j^{2n}(E_k, E_j)=0 \qquad m, n=1,2,3,\ldots. \tag{2.6}$$

By theorem 2.4, we have

$$(H, H_m)=0 \qquad m=1,2,\ldots. \tag{2.7}$$

Theorem 2.5. The Hamiltonian system defined by (2.8) is finite-dimensional and completely integrable in the Liouville sense.

$$p_{jt_m}=-\frac{\partial H_m}{\partial q_j} \qquad q_{jt_m}=\frac{\partial H_m}{\partial p_j} \qquad m=1,2,\ldots, \qquad j=1,2,\ldots,N \tag{2.8}$$

where H_m is defined by (2.5).

Proof. From (2.4), we have

$$(H_m, E_k)=0 \qquad k=1,2,\ldots,N$$

so that theorem 2.5 holds.

3. Relationship with the Levy equation hierarchy

Now, we consider the following eigenvalue problem:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}_x = \begin{bmatrix} -\frac{1}{2}(\lambda^2+u+v) & \lambda u \\ -\lambda & \frac{1}{2}(\lambda^2+u+v) \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \equiv M \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{3.1}$$

where u and v are called potentials, λ is the eigenparameter.

Let $\partial = \partial/\partial x$, $\partial\partial^{-1} = \partial^{-1}\partial = 1$, and the operators K and J take the following form:

$$K = \begin{bmatrix} -(\partial u + u\partial) & -\partial^2 - v\partial + \partial u \\ \partial^2 + u\partial - \partial v & \partial v + v\partial \end{bmatrix} \qquad J = \begin{bmatrix} 0 & \partial \\ \partial & 0 \end{bmatrix}. \tag{3.2}$$

We define Lenart's sequence $G_j, j = 0, 1, 2, \ldots$, by means of the recursion relations

$$KG_{j-1} = JG_j \qquad j = 0, 1, 2, \ldots, \qquad G_j = \begin{bmatrix} c_j \\ b_j \end{bmatrix} \qquad G_{-1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad j = 1, 2, \ldots. \tag{3.3}$$

Set

$$\bar{N}_j = \begin{bmatrix} \frac{1}{2}(\lambda^2+u+v)(c_j-b_j) - \frac{1}{2}(c_{jx}-b_{jx}) & \lambda(ub_j - uc_j - b_{jx}) \\ \lambda(c_j - b_j) & -\frac{1}{2}(\lambda^2+u+v)(c_j-b_j) + \frac{1}{2}(c_{jx}-b_{jx}) \end{bmatrix}. \tag{3.4}$$

Directly computing, we obtain

$$\begin{aligned} &\bar{N}_{jx} + \bar{N}_j M - M\bar{N}_j \\ &\quad = \begin{bmatrix} -\frac{1}{2} & \lambda \\ 0 & \frac{1}{2} \end{bmatrix} [(-\partial u - u\partial, -\partial^2 - v\partial + \partial u) G_j - \lambda^2 (0, \partial) G_j] \\ &\qquad + \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} [(\partial^2 + u\partial - \partial v, \partial v + v\partial) G_j - \lambda^2 (\partial, 0) G_j]. \end{aligned} \tag{3.5}$$

Further, we take

$$N_m = \sum_{j=0}^{m} \bar{N}_{j-1} \lambda^{2m-2j} \qquad m = 0, 1, 2, \ldots.$$

From (3.3), (3.4) and (3.5), we have the following theorem.

Theorem 3.1. The mth-order Levy equation

$$\begin{bmatrix} u \\ v \end{bmatrix}_{t_m} = JG_m = KG_{m-1} \tag{3.6}$$

is equivalent to the zero-curvature equation

$$M_{t_m} = N_{mx} - [M, N_m] = N_{mx} + N_m M - MN_m$$

which is the compatible condition for the following Lax pair:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}_x = M \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{3.1}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}_{t_m} = N_m \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}. \tag{3.7}$$

Example. By (3.3)

$$G_0=\begin{bmatrix} v \\ u \end{bmatrix} \qquad J^{-1}=\begin{bmatrix} 0 & \partial^{-1} \\ \partial^{-1} & 0 \end{bmatrix} \qquad G_1=J^{-1}KG_0=\begin{bmatrix} v_x-v^2+2uv \\ -u_x+u^2-2uv \end{bmatrix}$$

$$\begin{bmatrix} u \\ v \end{bmatrix}_{t_1}=\begin{bmatrix} (-u_x+u^2-2uv)_x \\ (v_x-v^2+2uv)_x \end{bmatrix}$$

has the Lax pair

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}_x=M\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}_{t_1}=\begin{bmatrix} -\frac{1}{2}(\lambda^2+u+v)(\lambda^2-v+u)-\frac{1}{2}(v_x-u_x) & \lambda(u^2-uv-u_x+\lambda^2 u) \\ -\lambda^3+\lambda(v-u) & \frac{1}{2}(\lambda^2+u+v)(u-v+\lambda^2)+\frac{1}{2}(v_x-u_x) \end{bmatrix} \times\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

Remark. From (3.2) and (3.6), let $\Phi=KJ^{-1}$, then

$$\Phi=\begin{bmatrix} -\partial-v+\partial u\partial^{-1} & -\partial u\partial^{-1}-u \\ \partial v\partial^{-1}+v & \partial+u-\partial v\partial^{-1} \end{bmatrix}$$

is the hereditary operator of the Levy hierarchy [16], and we have

$$\begin{bmatrix} u \\ v \end{bmatrix}_{t_m}=KG_{m-1}=KJ^{-1}JG_{m-1}=KJ^{-1}KG_{m-2}=\ldots=(KJ^{-1})^m JG_0=\Phi^m\begin{bmatrix} u_x \\ v_x \end{bmatrix}$$

so that equation (3.6) is called an mth-order Levy equation.

Now, let λ_j and $(p_j, q_j)^{\mathrm{T}}$ be the eigenvalue and eigenfunction of (3.1) under certain boundary conditions, say, periodic or decaying to zero at infinity

$$\begin{bmatrix} p_j \\ q_j \end{bmatrix}_x=\begin{bmatrix} -\frac{1}{2}(\lambda_j^2+u+v) & \lambda_j u \\ -\lambda_j & \frac{1}{2}(\lambda_j^2+u+v) \end{bmatrix}\begin{bmatrix} p_j \\ q_j \end{bmatrix} \qquad j=1,2,\ldots,N. \tag{3.1}$$

Then in the normal way we have [6]:

$$\operatorname{grad}\lambda_j=\begin{bmatrix} \delta\lambda_j/\delta u \\ \delta\lambda_j/\delta v \end{bmatrix}=\begin{bmatrix} p_j q_j-\lambda_j q_j^2 \\ p_j q_j \end{bmatrix}.$$

By (3.2), we obtain

$$K\begin{bmatrix} p_j q_j-\lambda_j q_j^2 \\ p_j q_j \end{bmatrix}=\lambda_j^2 J\begin{bmatrix} p_j q_j-\lambda_j q_j^2 \\ p_j q_j \end{bmatrix} \qquad j=1,2,\ldots,N. \tag{3.8}$$

Consider the following constraint [6]:

$$G_0=\begin{bmatrix} v \\ u \end{bmatrix}=\begin{bmatrix} \langle P, Q\rangle-\langle AQ, Q\rangle \\ \langle P, Q\rangle \end{bmatrix}. \tag{3.9}$$

From (3.3), (3.8) and (3.9), we have

$$c_j=\langle A^{2j}P, Q\rangle-\langle A^{2j+1}Q, Q\rangle \qquad b_j=\langle A^{2j}P, Q\rangle \qquad j=0,1,2,\ldots.$$

In the case of the constraint condition (3.9), the Lax pair (3.1), (3.7) of the mth-order Levy equation are nonlinearized respectively as follows:

$$p_{jx} = -\frac{\partial H}{\partial q_j} \qquad q_{jx} = \frac{\partial H}{\partial p_j} \qquad j = 1, 2, \ldots, N \tag{3.10}$$

$$p_{jt_m} = -\frac{\partial H_m}{\partial q_j} \qquad q_{jt_m} = \frac{\partial H_m}{\partial p_j} \qquad j = 1, 2, \ldots, N \tag{3.11}$$

where H is defined by (1.2), H_m is defined by (2.5).

According to theorem 2.4 and equations (2.6), (2.7), the Hamiltonian canonical systems (3.10) and (3.11) are completely integrable in the Liouville sense; and (3.10) and (3.11) are compatible [17]; therefore the Hamiltonian phase g_H^x and $g_{H_m}^{t_m}$ are commutable. Now, we arbitrarily choose an initial value $(P(0,0), Q(0,0))^{\mathrm{T}}$; set

$$\begin{bmatrix} P(x, t_m) \\ Q(x, t_m) \end{bmatrix} = g_H^x g_{H_m}^{t_m} \begin{bmatrix} P(0,0) \\ Q(0,0) \end{bmatrix} = g_{H_m}^{t_m} g_H^x \begin{bmatrix} P(0,0) \\ Q(0,0) \end{bmatrix} \tag{3.12}$$

then (3.12) is called an involutive solution of the Hamiltonian canonical systems (3.10) and (3.11). We thus obtain the following theorem.

Theorem 3.2. Suppose $(P(x, t_m), Q(x, t_m))^{\mathrm{T}}$ is an involutive solution of (3.10) and (3.11), then

$$v = \langle P(x, t_m), Q(x, t_m) \rangle - \langle AQ(x, t_m), Q(x, t_m) \rangle$$

$$u = \langle P(x, t_m), Q(x, t_m) \rangle$$

becomes the solution of the mth-order Levy equation (3.6).

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